

The Conley index for piecewise continuous maps

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Abstract

This paper gives the definition of the Conley index for a piecewise continuous map, which can be well defined on compatible isolating neighborhoods with Wàzewski property slightly weaker than continuous situation.

Key words: Conley index; piecewise continuous map; discontinuity; coding

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1 Introduction

The Conley index was first developed on flows [5], then it was extended to continuous maps [29, 33, 13]. The index has been extensively studied in recent years [26, 32, 20, 24].

The Conley index is an abstract extension of the Morse index, carrying both existence and stability information concerning a flow or a discrete dynamical system. By illustrating these properties, sufficient conditions for the presence of a bifurcation point for 1-dimensional flows were presented in [12, 10], and more generally, by using the general Conley index theory (see [31]), sufficient conditions for the existence of a bifurcation (sub-bifurcation) point of a family of flows on a compact metric space were also given in [12, 10]. There have been many successful applications of the Conley index, e.g., to the studies of travelling and shock waves [31], periodic solutions for Hamiltonian systems [6], infinite-dimensional semiflows [30], combustion problems [34], factoring problems [36], theory and computation for set-valued maps [17], rigorous numerics for dynamical systems [28, 7, 18], characterization of chaos or spatial-temporal chaos [21, 8], computer assisted rigorous proof for chaos [22, 27, 23, 25, 14], and so on. Therefore in the study of dynamical systems, including both flows and maps, the Conley index has been a very useful tool.

Dynamics of discontinuous maps is an emerging research field of dynamical systems [4, 35, 19, 2, 15, 3, 1, 11]. Discontinuities which may be caused by collision (impacting), switching, overflow (round-off), quantization etc., are unavoidable in both theory and applications. The main obstacle in studying discontinuous systems is the lack of necessary mathematical tools. In fact, due to discontinuities, most of analytical and topological methods cannot be directly used. Furthermore, huge number of discontinuous systems are even not measure-preserving, so ergodic theory may invalidate, too. Thus it is very important to develop new tools for discontinuous systems.

In this paper, we shall generalize the Conley index to discontinuous case. Usually, real world systems do not contain much discontinuous points, so here we only study piecewise continuous maps defined on finite partitions. We follow Goetz's construction [15, 16] to lift a discontinuous system (a piecewise continuous map) to a continuous one by graph of the coding map, then use the Conley index for continuous case (for example [29, 33, 13] etc.) to define the index of the discontinuous map. Due to discontinuity, one can imagine the index such defined may be fairly weak. But it does generalize continuous index and reserve some important information of the discontinuous system. The properties of the index here is more like the situation in [9], that is, the index is only defined on compatible isolating neighborhoods, and the Wàzewski property holds, while it is weaker than the usual one, such as the one given in [33]. Furthermore, due to the absence of continuity, which leads to the lack of suitable homotopy, the continuation property is lost.

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2 Preliminaries

We use \mathbb{R}^+ , \mathbb{Z} , \mathbb{Z}^+ , \mathbb{Z}^- , \mathbb{N} to denote the sets of nonnegative real numbers, integers, nonnegative integers, nonpositive integers and natural numbers, respectively. For a topological space X and a subset $A \subset X$, we shall denote by $\text{cl}_X(A)$, $\partial_X A$ and $\text{int}_X(A)$ the closure, boundary and the interior of A in X , and will omit the subscript if no confusion may be caused.

For a topological space X , a finite partition $\mathcal{A} = \{A_0, A_1, \dots, A_{n-1}\}$ ($n \in \mathbb{Z}^+$) means that (1) $A_i \subset X$ for any $i = 0, 1, \dots, n-1$; (2) $A_i \cap A_j = \emptyset$ whenever $i \neq j$; and (3) $\bigcup_{i=0}^{n-1} A_i = X$. Each A_i will be called a piece of the partition.

Suppose X, Y are two topological spaces. A map $f : X \rightarrow Y$ is called a finite piecewise continuous map from X to Y if there exists a finite partition $\mathcal{A} = \{A_0, \dots, A_{n-1}\}$ of X for some $n \in \mathbb{Z}^+$, such that $f_i := f|_{\text{cl}_X(A_i)} : \text{cl}_X(A_i) \rightarrow Y$ is continuous for each $i = 0, \dots, n-1$.

Given a piecewise continuous map f , we may coarsen the partition by unioning some of the pieces such that the new one is still piecewise continuous with the same map. Thus we can coarsen the partition finite times to get a new partition with minimum number of pieces, i.e., it cannot coarsen anymore, and call it the minimal partition with respect to f .

In the sequel, X will be assumed to be a locally compact metric space with metric d_X . We use PCM to denote piecewise continuous maps (with minimal finite partition) from locally compact metric space to itself, and call it piecewise continuous discrete dynamical system or piecewise continuous system for short.

For a PCM $f : X \rightarrow X$ with partition $\mathcal{A} = \{A_0, A_1, \dots, A_{n-1}\}$, the discontinuity set $D := \bigcup_{i \neq j} \text{cl}(A_i) \cap \text{cl}(A_j)$ should be carefully considered. We call a partition $\mathcal{B} = \{B_0, B_1, \dots, B_{m-1}\}$ of X an adjoint partition of \mathcal{A} if (1) $m = n$; (2) There exists a permutation ϕ of $\{0, 1, \dots, n-1\}$ such that $A_i - D = B_{\phi(i)} - D$ for every $i = 0, 1, \dots, n-1$; This is of course an equivalence relation, so we can say \mathcal{A}, \mathcal{B} are mutually adjoint. Now define $g : X \rightarrow X$ by $g|_{B_{\phi(i)}} := f_i|_{B_{\phi(i)}}$, then we get a new PCM with partition \mathcal{B} , and call it an adjoint map with f . Two adjoint maps may only differ on the discontinuity set.

3 Coding and continuous lifting

To study a PCM, coding is a very helpful method.

Definition 3.1. Let $f : X \rightarrow X$ be a PCM with partition $\mathcal{A} = \{A_0, \dots, A_{n-1}\}$. We define the coding map $\tau : X \rightarrow \Sigma_n^+ := \{0, 1, \dots, n-1\}^{\mathbb{Z}^+}$ by $\tau(x) = (k_0, k_1, \dots)$, where $x \in X$, $f^i(x) \in A_{k_i}$, $i \in \mathbb{Z}^+$ and $k_i \in \{0, \dots, n-1\}$.

Note that in the above definition, Σ_n^+ is exactly the one-sided symbolic sequence space with state space $\{0, 1, \dots, n-1\}$. We can give various product metric on it. For example, one can define $d_\Sigma : \Sigma_n^+ \times \Sigma_n^+ \rightarrow \mathbb{R}^+$ as $d_\Sigma(x, y) = \max\{\frac{1}{n+1} | x_n \neq y_n\}$, where x, y are two points in Σ_n^+ .

Remark 3.1. The coding map τ is not continuous.

Goetz indicated in his papers [15, 16] that any finite piecewise isometry can be lifted to a continuous map by using the graph of coding map. This process can be extended to piecewise continuous case.

Suppose $f : X \rightarrow X$ is a PCM and let $G_X := \{(x, \tau(x)) | x \in X\}$ be the graph of $\tau : X \rightarrow \Sigma_n^+$ topologized by the product metric $d_{G_X} : G_X \times G_X \rightarrow \mathbb{R}^+$, where

$$d_{G_X}((x_1, \tau(x_1)), (x_2, \tau(x_2))) = \max\{d_X(x_1, x_2), d_\Sigma(\tau(x_1), \tau(x_2))\}.$$

Then we have:

Proposition 3.1. The graph map $\tilde{f} : G_X \rightarrow G_X$, $(x, \tau(x)) \mapsto (f(x), \tau(f(x)))$ is continuous and the following diagram commutes:

$$\begin{array}{ccc} G_X & \xrightarrow{\tilde{f}} & G_X \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & X \end{array}$$

where π is the natural projection on the first component.

Proof. The proof is much the same as that in [15], though f here is more general. \square

Let $\text{cl}_{X \times \Sigma_n^+}(G_X)$ be the closure of G_X in $X \times \Sigma_n^+$, then we can naturally continuously extend $\tilde{f} : G_X \rightarrow G_X$ to $\hat{f} : \text{cl}(G_X) \rightarrow \text{cl}(G_X)$ by limit. That is, if $(x, s) \in \text{cl}(G_X) - G_X$ (then there must be a sequence $\{(x_n, \tau(x_n))\} \subset G_X$ such that $(x_n, \tau(x_n)) \rightarrow (x, s)$, as $n \rightarrow \infty$), thus we define $\hat{f}(x, s) := \lim_{n \rightarrow \infty} \tilde{f}(x_n, \tau(x_n))$. Unfortunately, for \hat{f} we do not have the commutative diagram as \tilde{f} has, that is, in general $\pi \circ \hat{f} \neq \hat{f} \circ \pi$.

From now on, if $A \subset X$, we use \tilde{A} to denote the graph of τ restrict on A , and \hat{A} to denote the closure of \tilde{A} in $\text{cl}(G_X)$.

For any compact subset $N \subset X$, \tilde{N} may not be a compact set in G . But for \hat{N} we have:

Lemma 3.2. *For any compact subset N of X , \hat{N} is compact in $\text{cl}(G_X)$.*

Proof. The result follows by noting that $N \times \Sigma_n^+$ is compact and \hat{N} is closed in it. \square

4 Invariant sets and isolating neighborhoods

Let $f : X \rightarrow X$ be a PCM and $A \subset X$, we define:

$\text{Inv}_{\tilde{f}}(A) := \{x \in A : \exists \{x_k\}_{k \in \mathbb{Z}^-} \subset A \text{ s.t. } x_0 = x \text{ and } f(x_k) = x_{k+1}, \forall k \in \mathbb{Z}^- - \{0\}\};$

$\text{Inv}_f(A) := \{x \in A : \exists \{x_k\}_{k \in \mathbb{Z}} \subset A \text{ s.t. } x_0 = x \text{ and } f(x_k) = x_{k+1}, \forall k \in \mathbb{Z}\}.$

Definition 4.1. *Let $f : X \rightarrow X$ be a PCM, a subset S of X is called an invariant set (with respect to f) if $\text{Inv}_f(S) = S$.*

Definition 4.2. *A compact set $N \subset X$ is an isolating neighborhood with respect to a PCM $f : X \rightarrow X$ if $\text{Inv}_f(N) \subset \text{int } N$.*

Definition 4.3. *An invariant set S is called an isolated invariant set with respect to a PCM $f : X \rightarrow X$ if there exists an isolating neighborhood N such that $S = \text{Inv}_f(N)$.*

Remark 4.1. *When $f : X \rightarrow X$ is continuous and $N \subset X$ is compact, $\text{Inv } N$ is also compact. But for a PCM, this is not always true. For example, consider $f : [0, 1] \rightarrow [0, 1]$, defined by $f(x) = x$ when $x \in [0, \frac{1}{2})$ and $f(x) = \frac{1}{2}x + \frac{1}{2}$ otherwise. Thus $[0, 0.6]$ is a compact set, while $\text{Inv}_f([0, 0.6]) = [0, \frac{1}{2}]$ is not compact in $[0, 1]$.*

Lemma 4.1. *Suppose $S \subset X$ is an invariant set with respect to PCM $f : X \rightarrow X$, then \tilde{S} is an invariant set in G_X with respect to $\tilde{f} : G_X \rightarrow G_X$.*

The proof is obvious.

Lemma 4.2. *Suppose $S \subset X$ is an isolated invariant set with respect to a PCM $f : X \rightarrow X$, then $\hat{S} \subset \text{cl}(G_X)$ is an invariant set with respect to $\hat{f} : \text{cl}(G_X) \rightarrow \text{cl}(G_X)$.*

Proof. We only need to show $\hat{S} \subset \text{Inv}_{\hat{f}}(\hat{S})$, since the inverse inclusion is obvious. For each $a \in \hat{S}$, there are only two cases:

Case 1: $a \in \tilde{S}$, then by Lemma 4.1, $a \in \text{Inv}_{\tilde{f}}(\tilde{S}) \subset \text{Inv}_{\tilde{f}}(\hat{S})$.

Case 2: $a = (x, u) \in \hat{S} - \tilde{S}$ for some $x \in \text{cl}(S)$. Then there must be a sequence $\{(x_n, \tau(x_n))\}_{n \in \mathbb{N}} \subset \tilde{S}$ s.t. $\lim_{n \rightarrow \infty} (x_n, \tau(x_n)) = (x, u)$. For each n , by Lemma 4.1, there exists $\{(x_n^{(i)}, \tau(x_n^{(i)}))\}_{i \in \mathbb{Z}} \subset \tilde{S}$, s.t. $\tilde{f}(x_n^{(i)}, \tau(x_n^{(i)})) = (x_n^{(i+1)}, \tau(x_n^{(i+1)}))$ and $(x_n^{(0)}, \tau(x_n^{(0)})) = (x_n, \tau(x_n))$. By the compactness of \hat{S} (S is isolated and Lemma 3.2), we may assume without loss of generality that $(x_n^{(i)}, \tau(x_n^{(i)})) \rightarrow (x^{(i)}, u^{(i)})$ for all i when $n \rightarrow \infty$. Therefore $\hat{f}(x^{(i)}, u^{(i)}) = \lim_{n \rightarrow \infty} \tilde{f}(x_n^{(i)}, \tau(x_n^{(i)})) = \lim_{n \rightarrow \infty} (x_n^{(i+1)}, \tau(x_n^{(i+1)})) = (x^{(i+1)}, u^{(i+1)})$ and $(x^{(0)}, u^{(0)}) = \lim_{n \rightarrow \infty} (x_n^{(0)}, \tau(x_n^{(0)})) = (x, u)$. \square

Recall that for \hat{f} defined in previous section, due to the lack of commutativity of the diagram, we should make our definition about isolating neighborhoods slightly more strict.

Definition 4.4. A compact set $N \subset X$ is a compatible isolating neighborhood with respect to PCM $f : X \rightarrow X$ if N is an isolating neighborhood and the intersection of ∂N and discontinuous set contains no points of $\text{Inv}_g^-(N)$, for any g adjoint with f .

Lemma 4.3. Suppose a compact subset $N \subset X$ is a compatible isolating neighborhood of PCM $f : X \rightarrow X$, then \hat{N} is an isolating neighborhood of \hat{f} .

Proof. By Lemma 3.2, \hat{N} is compact in $\text{cl}(G_X)$. It can be trivially seen that $\partial \hat{N}$ contains no points of $\text{Inv}_{\hat{f}}(\hat{N})$. Suppose $(x, s) \in \partial(\hat{N}) - \hat{N}$ and $(x, s) \in \text{Inv}_{\hat{f}}(\hat{N})$. Since $\hat{f} : \text{cl}(G_X) \rightarrow \text{cl}(G_X)$ is continuously extended by $\tilde{f} : G_X \rightarrow G_X$, we see that there is no point $(x, \tau(x)) \in G_X$ such that $\hat{f}(x, \tau(x)) = (x, s)$. Thus the backward trajectory of (x, s) are all contained in $\hat{N} - \hat{N}$, and one can easily check x is in $\text{Inv}_g^-(N)$ for some g adjoint with f . \square

5 The Conley index for continuous maps

In this section, we briefly recall the definition of the Conley index for a continuous map. There are various definitions, we choose here Szymczak's construction [33]. Since continuous dynamics can be viewed as PCM with just one piece, concepts such as isolating invariant sets and isolating neighborhoods defined in previous section also valid for continuous situation, so we will not restate these definitions.

To define the index, we require to introduce some categories.

Let \mathcal{K} denote an arbitrary category. $\text{Ob}(\mathcal{K})$ stands for its objects class, and $\text{Mor}_{\mathcal{K}}(A, B)$ means the morphism set from object A to object B . We use \mathcal{T} to denote topological space and continuous map category and \mathcal{T}' stands for its homotopy category.

Recall that the category $\text{Endo}(\mathcal{K})$ is defined by $\text{Ob}(\text{Endo}(\mathcal{K})) := \{(X, f) : X \in \text{Ob}(\mathcal{K}), f \in \text{Mor}_{\mathcal{K}}(X, X)\}$ and $\text{Mor}_{\text{Endo}(\mathcal{K})}((X, f), (X', f')) := \{\varphi \in \text{Mor}_{\mathcal{K}}(X, X') : \varphi \circ f = f' \circ \varphi\}$.

Szymczak introduced the category of objects equipped with a morphism over \mathcal{K} in [33], we call it the Szymczak category $\text{Sz}(\mathcal{K})$, which is defined as: the objects of $\text{Sz}(\mathcal{K})$ are the same as in $\text{Endo}(\mathcal{K})$ and

$$\text{Mor}_{\text{Sz}(\mathcal{K})}((X, f), (X', f')) := \left(\text{Mor}_{\text{Endo}(\mathcal{K})}((X, f), (X', f')) \times \mathbb{Z}^+ \right) / \sim$$

where $(\varphi_1, n_1) \sim (\varphi_2, n_2)$ if and only if there exists $k \in \mathbb{Z}^+$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f^{n_1+k}} & X \\ \downarrow f^{n_2+k} & & \downarrow \varphi_2 \\ X & \xrightarrow{\varphi_1} & X' \end{array}$$

The relation “ \sim ” defined above is an equivalence relation. We denote by $[\varphi, n]$ the morphism in $\text{Mor}_{\text{Sz}(\mathcal{K})}((X, f), (X', f'))$. The identity morphisms are given by $\text{id}_{(X, f)} = [\text{id}_X, 0]$.

We define the composition of two morphisms $[\varphi, n] \in \text{Mor}_{\text{Sz}(\mathcal{K})}((X, f), (X', f'))$ and $[\varphi', n'] \in \text{Mor}_{\text{Sz}(\mathcal{K})}((X', f'), (X'', f''))$ as $[\varphi', n'] \circ [\varphi, n] := [\varphi' \circ \varphi, n' + n]$.

Note that if $[g, n] \in \text{Mor}_{\text{Sz}(\mathcal{K})}((X, f), (X', f'))$, then $[g, n] = [g \circ f^k, k + n] = [f'^k \circ g, k + n]$ for each $k \in \mathbb{Z}^+$. In particular, if $m, n \in \mathbb{Z}^+$ and $h \in \text{Mor}_{\mathcal{K}}(X, X)$, then $[h^m, m] = [h^n, n] = [\text{id}_X, 0]$.

Now suppose $h : X \rightarrow X$ is a continuous map and S is an isolating invariant set, then there exists an index pair. That is a compact pair $P = (P_1, P_0)$ of X which satisfies (1) $S = \text{Inv}_h(\text{cl}(P_1 - P_0)) \subset \text{int}(P_1 - P_0)$; (2) $x \in P_0$ implies $h(x) \notin P_1 - P_0$; (3) $x \in P_1$, $h(x) \notin P_1$ implies $x \in P_0$. For the index pair P , we can induce a map on the quotient space $h_P : P_1/P_0 \rightarrow P_1/P_0$ defined by

$$h_P([x]) := \begin{cases} [h(x)], & x \in h^{-1}(P_1) \\ [P_0], & \text{otherwise} \end{cases}$$

and call it the index map. One can verify that the index map is continuous.

Szymczak's definition of the Conley index for an isolating invariant set S is the class of all objects isomorphic to $(P_1/P_0, h_P)$ in $\text{Sz}(\mathcal{T}')$, and it is independent upon the choice of P .

6 The Conley index for PCM's

Suppose $f : X \rightarrow X$ is a PCM and N is a compatible isolating neighborhood with respect to f , then by Lemma 4.3, \widehat{N} is also an isolating neighborhood of \widehat{f} . So $\text{Inv}_{\widehat{f}}(\widehat{N})$ meets an index pair $P = (P_1, P_0)$. Thus by definition, the index of $\text{Inv}_{\widehat{f}}(\widehat{N})$ is just the isomorphic class $[P_1/P_0, (\widehat{f})_P]$ in $\text{Sz}(\mathcal{T}')$. Since the Conley index for a continuous map is independent upon the choice of index pairs, we can give the following definition:

Definition 6.1. Let $f : X \rightarrow X$ be a PCM with minimal finite partition $\mathcal{A} = \{A_0, A_1, \dots, A_{n-1}\}$. If a compact set $N \subset X$ is a compatible isolating neighborhood, then we define the Conley index on N , denoted by $C(N)$ (or $C(N, f)$), to be the Conley index of $\text{Inv}_{\widehat{f}}(\widehat{N})$.

Remark 6.1. In continuous case, the Conley index is defined both on isolating invariant sets and isolating neighborhoods. But our definition of piecewise continuous case only define on compatible isolating neighborhoods. That is, for different compatible isolating neighborhoods with the same invariant set, their index may different.

Proposition 6.1 (Ważewski property). If $C(N)$ is nontrivial, then at least one of the following is true:

1. $\text{Inv}_f(N) \neq \emptyset$;
2. There exists some g adjoint with f ($g \neq f$), such that $\text{Inv}_g(N) \neq \emptyset$.

Proof. $C(N)$ is nontrivial implies the Conley index (continuous case) of \widehat{N} is nontrivial. Thus, by [33], $\text{Inv}_{\widehat{f}}(\widehat{N}) \neq \emptyset$. According to Lemma 4.2 we have $\widehat{\text{Inv}_f(N)} \subset \widehat{\text{Inv}_{\widehat{f}}(\widehat{N})} \neq \emptyset$. Suppose $\widehat{\text{Inv}_f(N)} \neq \emptyset$, then we get $\text{Inv}_f(N) \neq \emptyset$ and (1) is true. Otherwise, we assume $\widehat{\text{Inv}_f(N)} = \emptyset$, then there must exist a point $(x, s) \in \widehat{N} - \widetilde{N}$, such that $(x, s) \in \widehat{\text{Inv}_{\widehat{f}}(\widehat{N})}$. This means the forward and backward trajectory of (x, s) are both contained in \widehat{N} . In the proof of lemma 4.3, we have already seen that there is no point $(x, \tau(x)) \in \widetilde{N}$ such that $\widehat{f}(x, \tau(x)) = (x, s)$. So the backward trajectory of (x, s) is contained in $\widehat{N} - \widetilde{N}$. But for $i \in \mathbb{N}$, if $f^{i-1}(x, s) \in \widehat{N} - \widetilde{N}$, we have either $f^i(x, s) \in \widehat{N} - \widetilde{N}$ or $f^i(x, s) \in \widetilde{N}$. If for $i_0 \in \mathbb{N}$, $f^{i_0}(x, s) \in \widetilde{N}$, then $f^i(x, s)$ can never be back to $\widehat{N} - \widetilde{N}$ for $i > i_0$. Thus either forward trajectory of (x, s) is contained in $\widehat{N} - \widetilde{N}$ or there exists $i_0 \in \mathbb{N}$ such that $f^{i_0-1}(x, s) \in \widehat{N} - \widetilde{N}$ and $f^{i_0}(x, s) \in \widetilde{N}$. However both of these two cases imply (2) is true. \square

Remark 6.2. Other definitions of the Conley index for discrete semidynamical systems can be chosen to define PCM's index. If one choose the definition in [9], we even don't need to take closure of the graph of the coding map, but the index such defined can only detect positive invariant sets.

Example 6.1. Consider a piecewise continuous interval map $f : [-1, 2] \rightarrow [-1, 2]$ defined by

$$f(x) = \begin{cases} x + 1/3, & x \in [-1, -1/3) \\ x + 1, & x \in [-1/3, 0) \\ x + 2/3, & x \in [0, 1/3) \\ x - 1/3, & x \in [1/3, 2/3) \\ -x + 4/3, & x \in [2/3, 1) \\ x - 1, & x \in [1, 4/3) \\ x - 1/3, & x \in [4/3, 2] \end{cases}.$$

It can be easily verify that $N = [-1/3, 4/3]$ is an compatible isolating neighborhood, and $P = (\widehat{N}, \emptyset)$ is an index pair in $\text{cl}(G_X)$. So the index of N is the isomorphic class $[P_6, g]$ in $\text{Sz}(\mathcal{T}')$, in which P_6 represents the pointed six points space, and g is induced by \widehat{f}_P together with the homotopy equivalence from \widehat{N}/\emptyset to P_6 . It can be easily verified that the index is nontrivial. Meanwhile, according to the definition of f , we can see that $\text{Inv}[-1/3, 4/3]$ is nonempty. Thus our definition of the Conley index for PCM's does keep some existence information.

7 Conclusions

In this paper we lift a PCM to a continuous map, and then use the definition of the Conley index for a continuous map to define the index of a PCM, so we generalize the definition of the Conley index to some discontinuous systems. Just like the similar situation discussed in [9], where the Conley index for maps in absence of compactness was introduced, here the index is only defined on compatible isolating neighborhoods and the Wàzewski property is weaker than the usual index. However, it does keep some important information, as being shown by an example given in Section 6; moreover, for a continuous map the minimal finite partition contains only one piece, this implies that the lifting map is isomorphic to the original one in $\text{Endo}(\mathcal{T})$, hence the index we defined exactly generalize the continuous one. Unfortunately, due to the absence of continuity, which leads to the lack of suitable homotopy of PCMs, the continuation property of the Conley index is lost in piecewise continuous situation.

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